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DIVISORIAL MODULES AND KRULL MORPHISMS

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Introduction

Let $R \subsetneq S$ be an inclusion of Krull domains. The extension S/R is said to satisfy condition (PDE), and the inclusion of R in S is said to be a Krull morphism, if for every height one prime q of S, $q \cap R$ has height at most one. Condition (PDE) is also called (NBU). This condition is useful because when it holds there is an induced map from the class group of R to that of S (see [3, §6]). In this paper we obtain a module-theoretic characterization of when S/R satisfies (PDE), namely precisely when S is divisorial as an R-module. In the course of obtaining this characterization we describe how divisoriality relates to flatness.

Divisoriality, flatness and condition (PDE)

Let R be a Krull domain, K its field of fractions, Z the set of prime ideals of height one in R. Let M be a torsion-free R-module and let $V = K \otimes_R M$. We define the R-submodule $\tilde{M}$ of V by the formula

$$\tilde{M} = \bigcap_{p \in Z} M_p.$$

There is an obvious inclusion of M into $\tilde{M}$, and if $M = \tilde{M}$ we say M is *divisorial.*

Let M be a torsion-free R-module, $V = K \otimes_R M$. M is an *R-lattice* if there is an R-module F of finite type with $M \subsetneq F \subsetneq V$. It is easy to verify that rank(M) is then finite and that F may be taken to be a *free* R-module of finite type. If M is an R-lattice so is

$$(R : M) = \{ f \in \mathrm{Hom}_K(V, K) \mid f(M) \subseteq R \},$$

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and there is a natural isomorphism

$$\tilde{M} \simeq (R:(R:M)).$$

Since $(R:M)$ has a natural identification with $\mathrm{Hom}_R(M,R)$, it follows that an R-lattice M is divisorial if and only if M is reflexive, i.e. if and only if the canonical homomorphism $M \to M^{**}$ is an isomorphism. For details see [1, Ch. VIII, §4, no. 2], [2, Ch. III, §8], or [3, §2].

For p in Z the ring R_p is a principal ideal domain. Hence, an R_p-module is R_p-flat if and only if it is torsion-free. Since R_p is a flat R-module, any R_p-module that is R_p-flat is R-flat. Hence, any divisorial R-module M is an intersection in V (its extension to K) of flat R-modules M_p. The family $\{M_p\}_{p\in Z}$ has a finiteness property with respect to V, namely that each element V is in all but a finite number of the M_p. This follows from R being a Krull domain, specifically from the fact that each non-zero element of R is a unit in all but finitely many R_p, p in Z.

Let V be a vector space over K, and $\{M_i\}_{i\in I}$ a family of R-submodules of V, with $KM_i = V$ for each i in I. We shall say this family is of *finite character* if each element v of V is in all but finitely many M_i.

Lemma 1. *Let $\{M_i\}_{i\in I}$ be a family of finite character. Let S be a multiplicative subset of R. Then*

(a) $S^{-1}(\bigcap_{i\in I} M_i) = \bigcap_{i\in I} S^{-1}M_i$.

(b) *If each M_i is a divisorial R-module, so is $\bigcap_{i\in I} M_i$.*

Proof. (a) The left-hand term is clearly contained in the right-hand one. Let v be in the right-hand term. By the finite character of the family $\{M_i\}$, there is a finite subset J of I such that v is in M_j for j in L, the complement of J in I. It is generally true that

$$S^{-1}(M \cap N) = (S^{-1}M) \cap (S^{-1}N) \tag{1}$$

for any R-modules M and N. It follows that

$$S^{-1}\left(\bigcap_{j\in J} M_j\right) = \bigcap_{j\in J} S^{-1}M_j.$$

By hypothesis, v is in the right-hand side of the last formula, hence in the left hand-side. Letting $M = \bigcap_{j\in J} M_j$, $N = \bigcap_{i\in L} M_i$ and applying (1) we conclude v is in the left-hand term of (a).

(b) follows by using (a) as necessary with $S = R - p$, p in Z, and the definition of divisoriality.

Proposition 1. Let *R be a Krull domain with field of fractions K. Let M be a torsion-free R-module, $V = K \otimes_R M$. Then M is divisorial if and only if M is the intersection of a family of finite character each module of which is R-flat.*

Proof. This follows from the lemma above and the discussion preceding it.

Corollary 1. *Let R be a Krull domain, M a torsion-free R-module. Then $\tilde{M}$ is divisorial.*

In the proof of Lemma 1 we exploited the fact that

$$S^{-1}(M\cap N)=S^{-1}M\cap S^{-1}N$$

to show that

$$S^{-1}\left(\bigcap_i M_i\right)=\bigcap_i S^{-1}M_i$$

when $\{M_i\}_{i\in I}$ is a family of finite character. The same idea can be used to prove that for such a family

$$M\otimes_R\left(\bigcap_i M_i\right)=\bigcap_i (M\otimes_R M_i)$$

when M is a flat R-module, since tensoring with a flat module preserves finite intersections [1,Ch. I, §2, no. 6]. We record this result for later reference.

Corollary 2. *Let M be a flat R-module, and $\{M_i\}_{i\in I}$ a family of finite character. Then*

$$M\otimes_R\left(\bigcap_{i\in I} M_i\right)=\bigcap_{i\in I} (M\otimes_R M_i).$$

Corollary 3. *Let S be a multiplicative set of R and M an R-module. If M is divisorial over R then $S^{-1}M$ is divisorial over $S^{-1}R$.*

Let M and N be torsion-free R-modules. let MN denote the image of $M\otimes_R N$ in $K\otimes_R M\otimes_R N$. Define the *modified tensor product* of M and N by

$$M\tilde{\otimes}_R N=\widetilde{MN}.$$

For (m,n) in $M\times N$ let $\alpha(m,n)$ be the element $1\otimes m\otimes n$ of $K\otimes_R M\otimes_R N$. View α as a map to $M\tilde{\otimes}_R N$. Let γ be the map sending (m,n) to $m\otimes n$ in $M\otimes_R N$. Because α is bilinear there is a commutative diagram

$$\begin{array}{ccc} & M\times N & \\ {}^{\beta}\swarrow & & \searrow^{\alpha} \\ M\otimes_R N & \xrightarrow[\gamma]{} & M\tilde{\otimes}_R N \end{array} \tag{2}$$

We record some basic facts about $\tilde{\otimes}_R$, some of which were noted by Yuan in Lemma 4 of [4] for modules of finite type and R noetherian.

Proposition 2. *Let L, M, N, M_i be torsion-free R-modules. Then:*

(a) *Given an R-homomorphism $f: M \to N$, there exists a unique R-homomorphism $\tilde{f}: \tilde{M} \to \tilde{N}$ which on M restricts to f. For $g: L \to M$ we have $(\widetilde{fg}) = \tilde{f}\tilde{g}$.*

(b) *$M \tilde{\otimes}_R N$ is divisorial.*

(c) *If L is divisorial and $\delta: M \times N \to L$ is an R-bilinear map, then there exists a unique R-homomorphism $\lambda: M \tilde{\otimes}_R N \to L$ satisfying $\lambda\alpha = \delta$ (with α as in (2) above).*

(d) *If M and N are R-lattices so is $M \tilde{\otimes}_R N$.*

(e) *If M is R-flat then it is divisorial. If in addition N is divisorial then the map γ of (2) above is an isomorphism.*

(f) *If B is an R-algebra and M is a B-module then there is a B-module structure on $M \tilde{\otimes}_R N$ which makes γ a B-module homomorphism.*

(g) *$R \tilde{\otimes}_R M = M$.*

(h) *$L \tilde{\otimes}_R (M \tilde{\otimes}_R N) = (L \tilde{\otimes}_R M) \tilde{\otimes}_R N$.*

(i) *$L \tilde{\otimes}_R (\bigoplus_i M_i) = \bigoplus_i (L \tilde{\otimes}_R M_i)$.*

(j) *$M \tilde{\otimes}_R N = N \tilde{\otimes}_R M$.*

(k) *Let S be a multiplicative subset of R and let $B = S^{-1}R$. Then $S^{-1}(M \tilde{\otimes}_R N) = S^{-1}M \tilde{\otimes}_B S^{-1}N$.*

Proof. (a) is clear. (b) follows from the corollary to Proposition 1. (c) follows from (a) and (b). (d) follows from the well-known facts that if M, N are R-lattice, so is MN and therefore $(R:(R:MN))$ is an R-lattice as well.

The first assertion of (e) is a consequence of Proposition 1. The second assertion follows from Corollary 2 to Proposition 1. (f) follows from (b) and (c). Assertions (g) to (j) are easy to prove.

To prove (k) we shall establish that there are maps in both directions between $S^{-1}(M \tilde{\otimes}_R N)$ and $S^{-1}M \tilde{\otimes}_B S^{-1}N$ whose composites are clearly the identity maps. First note that each of the modules involved is divisorial over B. For $S^{-1}(M \tilde{\otimes}_R N)$ this is true by (b) and by Corollary 3 to Proposition 1. For $S^{-1}M \tilde{\otimes}_B S^{-1}N$ we need only invoke (b) with R replaced by $S^{-1}R$. The existence of the maps we want is now easily established using (c) and properties of the functor $S^{-1}(\ )$.

Corollary. *Let R be a Krull domain, M an R-lattice. If M is flat it is a projective R-module of finite type.*

Proof. There is a natural map f from $M \otimes_R M^*$ to $\mathrm{End}_R(M)$ satisfying $f(x \otimes \alpha) = \alpha(\ )x$ for x in M, α in M^*. Because M is a divisorial R-lattice, so is $\mathrm{End}_R(M)$ [3, Proposition 2.6]. $M \otimes_R M^*$ is a divisorial R-module by (b) and (e) above. For each height one prime p of R, f_p is an isomorphism (use (k) above and the equality $\mathrm{Hom}_R(M,R)_p = \mathrm{Hom}_{R_p}(M_p, R_p)$ (see [2, p. 151, Cor. 8.4] or [3, Cor. 5.5]). By divisoriality of the modules involved, f must itself be an isomorphism. If $f(\sum x_i \otimes \alpha_i) = 1$ then the finite set $\{x_i, \alpha_i\}$ is a projective basis for M.

Let $R \subseteq S$ be an inclusion of Krull domains having respective fraction fields K and L. Let M be an R-lattice, $V = K \otimes_R M$. Then SM is an S-lattice in $L \otimes_S SM$ [3, Proposition 2.2 (v)]. Let H be a free S-module of finite rank, with $SM \subsetneq H \subsetneq L \otimes_S SM$. Suppose S is divisorial as an R-module. Then H is also divisorial as an R-module, because it is free over S. It follows in this case that $SM \subseteq \widetilde{SM} \subsetneq H$ (the construction $\tilde{}$ is with respect to R). SM is an S-module by (f) of Proposition 1, hence $\widetilde{SM}$ is also an S-lattice. We have proved:

Lemma 2. *Let $R \subsetneq S$ be an inclusion of Krull domains. Suppose S is divisorial as an R-module. If M is an R-lattice, $S \tilde{\otimes}_R M$ is an S-lattice.*

Proposition 3. *Let $R \subseteq S$ be an inclusion of Krull domains. The following conditions are then equivalent:*

(1) *For M any divisorial R-lattice, $S \tilde{\otimes}_R M$ is a divisorial S-lattice.*

(2) *S is divisorial as an R-module.*

Proof. Our proof follows that given by Yuan in the noetherian case [4, Proposition 2]. Assuming (1), $\tilde{S}$ must be a divisorial S-lattice. Let L be the field of fractions of S. Then $\tilde{S} \subseteq L$. Then for any x in $\tilde{S}$ there is an S-module H of finite type such that $\sum Sx^i \subseteq H$. Because S is a Krull domain it is completely integrally closed. This implies x is in S, hence $\tilde{S} = S$ and (2) holds.

Suppose (2) holds. Let M be a divisorial R-lattice. By Lemma 2, $S \tilde{\otimes}_R M$ is an S-lattice. Let Y denote the set of height one primes of S and in the intersections below let p range over Z, q over Y.

$$
\begin{aligned}
\bigcap_q (S \tilde{\otimes}_R M)_q &= \bigcap_q \left(S_q \otimes_S \left(\bigcap_p (SM)_p \right) \right) \\
&= \bigcap_q \bigcap_p (S_q \otimes_S (SM)_p) \quad \text{(Cor. 2 to Prop. 1)} \\
&= \bigcap_q \bigcap_p S_q \otimes_S S \otimes_R M_p \quad ((SM)_p = S \otimes_R M_p \text{ since } M_p \text{ is } R\text{-flat}) \\
&= \bigcap_p \bigcap_q S_q \otimes_R M_p \\
&= \bigcap_p \left(\bigcap_q S_q \right) \otimes_R M_p \quad \text{(Corollary 2 again)} \\
&= \bigcap_p S \otimes_R M_p \quad (S \text{ is a Krull domain}) \\
&= S \tilde{\otimes}_R M \quad \text{(by definition).}
\end{aligned}
$$

This shows $S \tilde{\otimes}_R M$ is divisorial as an S-module and completes the proof.

Corollary 1 (of the proof). *Let $R \subsetneq S$ be Krull domains, with S divisorial as an R-module. If M is a divisorial R-module, $S \tilde{\otimes}_R M$ is a divisorial S-module.*

Corollary 2. *Let $R \subseteq S \subseteq T$ be inclusions of Krull domains. If S is divisorial as an R-module and T is flat as an S-module then T is divisorial as an R-module.*

Proof. Because T is S-flat, $T \tilde{\otimes}_S Q = T \otimes_S Q$ for Q any S-module ((e) of Proposition 2). Then for any R-lattice M

$$\begin{aligned} T \tilde{\otimes}_S (S \tilde{\otimes}_R M) = T \otimes_S (S \tilde{\otimes}_R M) &= T \otimes_S \left(\bigcap_{p \in Z} S \otimes_R M_p \right) \\ &= \bigcap_{p \in Z} T \otimes_S S \otimes_R M_p \quad \text{(Cor. 2 to Prop. 1)} \\ &= \bigcap_{p \in Z} T \otimes_R M_p \\ &= T \tilde{\otimes}_R M. \end{aligned}$$

If M is a divisorial R-lattice, $S \tilde{\otimes}_R M$ is a divisorial S-lattice and then $T \tilde{\otimes}_S (S \tilde{\otimes}_R M)$ is a divisorial T-lattice (Proposition 3). The equality above then shows $T \tilde{\otimes}_R M$ is a divisorial T-lattice. By Proposition 3, T is divisorial as an R-module.

Corollary 3. *Let $R \subseteq S$ be an inclusion of Krull domains, and assume S is a flat R-module. Then for M a divisorial R-lattice, $S \otimes_R M$ is a divisorial S-lattice.*

Proof. A consequence of Proposition 3 and (e) of Proposition 2.

Lemma 3. *Let $R \subseteq S$ be an inclusion of Krull domains. Let M be an S-module. Suppose S/R satisfies the condition* (PDE), *and that M is divisorial over S. Then M is divisorial over R.*

Proof. We know $M = \bigcap M_q$, as q ranges over the height one primes of S. If we can show that each M_q is R-flat, we can conclude M is divisorial by Proposition 1. We have that M_q is flat as an S_q-module since S_q is a principal ideal domain. Let $p = q \cap R$. Because (PDE) holds, R_p is a D.V.R. (perhaps a field). Thus S_q is flat over R_p, hence over R. Hence M_q is flat over R.

Theorem 1. *Let $R \subseteq S$ be an inclusion of Krull domains. Then S/R satisfies* (PDE) *if and only if S is divisorial as an R-module.*

Proof. If (PDE) holds, Lemma 3 implies the extension is divisorial. Suppose conversely that S is divisorial as an R-module. Let q be a height one prime of S, and suppose that $P = q \cap R$ has height greater than one. First we will assume $q = Sz$ for some z in S. We will show that height$(P) > 1$ implies $1/z$ is in S_p for all p of height one in R, but is clearly not in S, contradicting divisoriality of S as an R-module. To show $1/z$ is in S_p, note that $p \neq P$. Let b be in P but not in p. But b is in q, hence $b = sz$ with s in S. Then $1/z = s/b$ with b in $R - p$, so $1/z$ is in S_p.

We can reduce to the case where q is principal as follows. S_q, R_P are Krull domains, with $R_P \subsetneq S_q$ and $PR_P = R_P \cap qS_q$. Thus if (PDE) does not hold for S/R, it does not hold for S_q/R_P. Moreover, S_q is flat over S_P because it is a localization. But S divisorial over R implies S_P is divisorial over R_P (Corollary 3 to Proposition 1). Thus $R_P \subsetneq S_P \subsetneq S_q$ are Krull domains satisfying the hypotheses of Corollary 1 to Proposition 3. By that corollary we conclude S is divisorial over R_P. Since S_q is a D.V.R., qS_q is principal, and we complete our proof by referring to the first case treated above.

Corollary 1. *Let $R \subsetneq S$. Let M be an S-module. If S is divisorial as an R-module and M is divisorial as an S-module then M is divisorial as an R-module.*

Proof. An immediate consequence of Lemma 3 and Theorem 1.

Corollary 2. *Let $R \subsetneq S \subsetneq T$ be inclusions of Krull domains, with S divisorial as an R-module and T divisorial as an S-module. Let M be a divisorial R-module. Then $T \tilde{\otimes}_R M \simeq T \tilde{\otimes}_S (S \tilde{\otimes}_R M)$.*

Proof. By the previous corollary T is divisorial as an R-module. By Corollary 1 of Proposition 3, $T \tilde{\otimes}_R M$ is divisorial as a T-module. So is $T \tilde{\otimes}_S (S \tilde{\otimes}_R M)$. There is a natural map R-homomorphism, call it f, from the first of these modules to the second (use (a) of Proposition 2). For each height one prime p of R, f_p is an isomorphism (this follows from (k) and (e) of Proposition 2 and the relation $T \otimes_R M \simeq T \otimes_S (S \otimes_R M)$.

Let $R \subsetneq S$ be Krull domains, M fand R-module. Define $S \otimes_R' M = \bigcap_{q \in Y} (S \otimes_R M)_q$, with Y the set of height one primes of S.

Proposition 4. *Let $R \subsetneq S$ be an extension of Krull domains. Then S is divisorial as an R-module if and only if for all divisorial R-modules M we have $S \otimes_R' M = S \tilde{\otimes}_R M$.*

Proof. Let S be divisorial over R. Let $N = SM$, $N' = S \otimes_R' M$ and $\tilde{N} = S \tilde{\otimes}_R M$. We have $N \subsetneq N'$ hence $N_p \subsetneq N_p'$ for p in Z, hence

$$\tilde{N} \subsetneq \bigcap_{p \in Z} N_p'. \tag{3}$$

But N' is a divisorial S-module by construction, hence is a divisorial R-module by Corollary 1 to Theorem 1. It follows from this and (3) that $\tilde{N} \subsetneq N'$. Similarly $\tilde{N}$ is a divisorial S-module (the proof of Proposition 3 shows that for M a divisorial R-module $S \tilde{\otimes}_R M$ is a divisorial S-module), and this leads to the inclusion $\tilde{N} \subsetneq N'$. Thus $\tilde{N} = N'$.

Conversely, if $N' = \tilde{N}$ for $N = SM$ and M a divisorial R-module, taking $M = R$ shows that $\tilde{S} = S'$. But S is a Krull domain, so $S' = S$, hence $S = \tilde{S}$, and S is divisorial as an R-module.

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